

NIELSEN EQUIVALENCE IN FUCHSIAN GROUPS AND SEIFERT FIBERED SPACES

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§0. INTRODUCTION

IN THIS paper we investigate Heegaard splittings Σ of Seifert fibered 3-manifolds M and generating systems for the underlying Fuchsian groups G . We introduce and compute a numerical invariant $\mathcal{N}(\Sigma) \in \mathbb{R}^+$, which, in the case where the exceptional fibers of M have odd and relatively prime multiplicities, completely classifies “vertical” Heegaard splittings of minimal genus. In particular $\mathcal{N}(\Sigma)$ distinguishes between minimal genus “vertical” Heegaard splittings of Seifert fibered homology spheres. The invariant $\mathcal{N}(\Sigma)$ is derived from a numerical invariant with values in \mathbb{R}^+ which exhibits Nielsen inequivalent (see Definition 1.1) generating systems of minimal cardinality for G . We prove:

THEOREM 1. *Let G be a Fuchsian group with presentation*

$$G = \langle q_1, \dots, q_m, a_1, b_1, \dots, a_g, b_g | q_i^{\alpha_i} (i = 1, \dots, m), q_1 \dots q_m \prod_{i=1}^g [a_i, b_i] \rangle$$

with $m \geq 3$ or $g \geq 1$ and odd and pairwise relatively prime exponents $\alpha_i > 1$. Two generating systems of G , $\mathcal{U} = \{q_1^{v_1}, \dots, q_{j-1}^{v_{j-1}}, q_j^{v'_j+1}, \dots, q_m^{v_m}, a_1, b_1, \dots, a_g, b_g\}$ and $\mathcal{V} = \{q_1^{v'_1}, \dots, q_k^{v'_k-1}, q_k^{v'_k+1}, \dots, q_m^{v'_m}, a_1, b_1, \dots, a_g, b_g\}$ with $v_i, v'_i \in \mathbb{Z}$, $1 \leq v_i, v'_i \leq \alpha_i/2$, $\text{g.c.d.}(v_i, \alpha_i) = \text{g.c.d.}(v'_i, \alpha_i) = 1$, are Nielsen equivalent if and only if;

- (i) $j = k$ and $v_i = v'_i$, or
- (ii) $j \neq k$ and $v_k = 1, v'_j = 1$, and $v_i = v'_i$ for $i \neq j, k$.

In light of a theorem of Rosenberger (see [16], Satz 2.2) Theorem 1 gives a complete classification, up to Nielsen equivalence, of all $2g + m - 1$ element generating systems of G when $g \geq 1$ and $m \geq 2$. The proof of Theorem 1 is based on the fact that for certain representations $\rho: \mathbb{Z}G \rightarrow \mathbb{M}_2(\mathbb{C})$, from the group ring of G into the (2×2) -matrices over \mathbb{C} , the Jacobian matrix of Fox derivatives $\rho(\partial \mathcal{V} / \partial \mathcal{U})$ is well defined. Such representations ρ are provided by the natural representations of the Fuchsian group G in $\text{PSL}_2(\mathbb{C})$. In particular the value $\det \rho(\partial \mathcal{V} / \partial \mathcal{U})$ turns out to be a positive real number which depends only on the Nielsen equivalence classes of the generating systems \mathcal{U} and \mathcal{V} . Its computation shows the remarkable fact that it is independent of the particular choice of the representation ρ .

For each Seifert fibered space M with $m \geq 2$ exceptional fibers, we exhibit $2^{m-1} - 1$ “vertical” (see Definition 2.4) Heegaard splittings of genus $2g + m - 1$. These Heegaard splittings will be minimal except in one special case (see [5], Theorem 1.1). In each

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Heegaard splitting Σ of M , one can choose cores of the handlebodies to obtain generating systems for $\pi_1(M)$. In particular one gets generating systems \mathcal{U} and \mathcal{V} of the quotient group $\pi_1(M)/\text{center} \cong G$, which is a Fuchsian group as above. The Nielsen equivalence classes of \mathcal{U} and \mathcal{V} are independent of the choice of the cores in each handlebody H_i . We denote $\det \rho(\partial \mathcal{V} / \partial \mathcal{U})$ by $\mathcal{N}(\Sigma)$. Since we regard two Heegaard splittings which arise from interchanging the handlebodies as equal, $\mathcal{N}(\Sigma)$ is determined only up to an exponent ± 1 . We resolve this ambiguity by normalizing so that $0 < \mathcal{N}(\Sigma) \leq 1$.

Two isotopic (homeomorphic) Heegaard splittings Σ_1 and Σ_2 of M give rise to two Nielsen equivalent pairs (up to an automorphism of G , resp.) of generating systems $\mathcal{U}_1, \mathcal{V}_1$ and $\mathcal{U}_2, \mathcal{V}_2$ of G . We obtain.

THEOREM 2. *Let M be a Seifert fibered space with Seifert invariants $\{g, e | (x_1, \beta_1), \dots, (x_m, \beta_m)\}$, $m = 2$ and $g > 0$ or $m \geq 3$, and all x_i odd and pairwise relatively prime. Let Σ_1 and Σ_2 be two vertical Heegaard splittings of genus $2g + m - 1$ of M . If $\mathcal{N}(\Sigma_1) = \mathcal{N}(\Sigma_2)$ then Σ_1 is isotopic to Σ_2 and if $\mathcal{N}(\Sigma_1) \neq \mathcal{N}(\Sigma_2)$ then Σ_1 and Σ_2 are non-homeomorphic.*

In particular, if M is a homology sphere which has m exceptional fibers with all x_i odd and $\beta_i \neq \pm 1 \pmod{x_i}$, then M has $2^{m-1} - 1$ non-homeomorphic vertical Heegaard splittings of genus $m - 1$.

The method of distinguishing Heegaard splittings by distinguishing Nielsen equivalence classes of corresponding generators was used in [15] and [2] to get a classification of genus 2 Heegaard splittings of Seifert fibered spaces over S^2 with three exceptional fibers. However in both papers the authors use the fact that, in a group of rank 2, the conjugacy class of the commutator of the generators is an invariant up to inversion of the Nielsen equivalence class of the generating set. Unfortunately this fact has no analogue in groups of higher rank (see [12], p. 44). In fact the method presented in §1 to determine Nielsen equivalence classes of generating systems of G is quite different from the other methods used in the literature (part of it has been suggested in [14]). Compare also with [1].

§1. NIELSEN EQUIVALENT GENERATING SYSTEMS OF FUCHSIAN GROUPS

Let G be a group with ordered generating systems $\mathcal{U} = \{u_1, \dots, u_n\}$ and $\mathcal{V} = \{v_1, \dots, v_n\}$ and let F_n denote the free group of rank n .

Definition 1.1. The generating systems \mathcal{U} and \mathcal{V} of G will be called *Nielsen equivalent* if there is an epimorphism $\varphi: F_n \twoheadrightarrow G$ and bases $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ of F_n such that $\varphi(x_i) = u_i$ and $\varphi(y_i) = v_i$ for $i = 1, \dots, n$.

Denote by $\partial/\partial x_i: \mathbb{Z}F_n \rightarrow \mathbb{Z}F_n$ the i -th Fox derivative of the integer group ring $\mathbb{Z}F_n$, i.e. the unique \mathbb{Z} -linear function satisfying $\partial x_i / \partial x_j = \delta_{i,j}$ for each i, j and for $a, b \in F_n$ $\partial(ab) / \partial x_i = \partial a / \partial x_i + a \partial b / \partial x_i$. Two bases $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ of F_n give rise to a Jacobian matrix $(\partial y_i / \partial x_j)_{i,j}$ with entries in $\mathbb{Z}F_n$ (see [7], p. 123). As X and Y are bases of F_n they are related by a finite sequence of elementary Nielsen transformations (see [13], p. 130). An application of the chain rule property of the Fox derivative (see [7], p. 125) to this sequence determines a decomposition of $(\partial y_i / \partial x_j)_{i,j}$ into a finite product of elementary matrices over $\mathbb{Z}F$. By elementary matrices we mean $\mathbb{Z}F_n$ -matrices with units $\pm w \in F_n$ on the diagonal and at most one off diagonal non zero element. The epimorphism $\varphi: F_n \twoheadrightarrow G$ induces a natural ring homomorphism $\mathbb{Z}F_n \rightarrow \mathbb{Z}G$, also denoted by φ . It follows that the matrix $\varphi(\partial y_i / \partial x_j)_{i,j} \in \mathbb{M}_n(\mathbb{Z}G)$ is invertible and can be decomposed into a finite product of elementary $\mathbb{Z}G$ -matrices.

To simplify the exposition we will assume, without loss of generality, that F_n is free on $X = \{x_1, \dots, x_n\}$. Now let v be an element of G and y, y' be elements of F_n that satisfy $\varphi(y) = \varphi(y') = v$. If $\ker \varphi$ is normally generated by $r_1, \dots, r_s \in F_n$, then $y' = \left(\prod_{k=1}^t w_k s_k w_k^{-1} \right) y$ for some $w_i \in F_n$ and $s_k \in \{r_1^{\pm 1}, \dots, r_s^{\pm 1}\}$. We obtain:

LEMMA 1.2. *Let v_1, \dots, v_n be elements of G and y_i, y'_i elements of $F_n = F(x_1, \dots, x_n)$ such that $\varphi(y_i) = \varphi(y'_i) = v_i$. Then the matrix $(\varphi(\partial y_i / \partial x_j) - \varphi(\partial y'_i / \partial x_j))_{1 \leq i, j \leq n}$, has entries in the left ideal of $\mathbb{Z}G$ generated by $\{\varphi(\partial r_i / \partial x_j) | i = 1, \dots, s, j = 1, \dots, n\}$.*

Proof. Applying the product rule for Fox derivatives to a typical expression $\left(\prod_{k=1}^t w_k s_k w_k^{-1} \right) y_i$, (for $s_k \in \{r_1^{\pm 1}, \dots, r_s^{\pm 1}\}$) we get:

$$\begin{aligned} \varphi\left(\frac{\partial y_i}{\partial x_j}\right) &= \varphi\left(\frac{\prod_{k=1}^t w_k s_k w_k^{-1}}{\partial x_j}\right) + \varphi\left(\prod_{k=1}^t w_k s_k w_k^{-1}\right) \varphi\left(\frac{\partial y_i}{\partial x_j}\right) \\ &= \sum_{k=1}^t \left[\varphi\left(\frac{\partial w_k}{\partial x_j}\right) + \varphi(w_k) \varphi\left(\frac{\partial s_k}{\partial x_j}\right) + \varphi(w_k) \varphi\left(\frac{\partial w_k^{-1}}{\partial x_j}\right) \right] + \varphi\left(\frac{\partial y_i}{\partial x_j}\right) \\ &= \sum_{k=1}^t \varphi(w_k) \varphi\left(\frac{\partial s_k}{\partial x_j}\right) + \varphi\left(\frac{\partial y_i}{\partial x_j}\right). \end{aligned} \quad \square$$

Remark 1.3. In standard homological terminology, (see [6], p. 43), any generating system $\mathcal{U} = \{u_1, \dots, u_n\}$ for the group G defines an epimorphism $\varphi: F(x_1, \dots, x_n) \rightarrow G$, $\varphi(x_i) = u_i$, which gives rise to an exact sequence of left $\mathbb{Z}G$ -modules:

$$0 \rightarrow N_{\mathcal{U}} \rightarrow \mathbb{Z}G\langle \mathcal{X}_1, \dots, \mathcal{X}_n \rangle \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0.$$

Here ε is the augmentation map, ∂_1 is given by $\partial_1(\mathcal{X}_i) = u_i - 1$, and the relation module $N_{\mathcal{U}}$ is isomorphic to $\ker \varphi / [\ker \varphi, \ker \varphi]$. The above Lemma 1.2 amounts to showing that $N_{\mathcal{U}}$ is generated as a left $\mathbb{Z}G$ -module by the elements $\sum_{j=1}^n \varphi(\partial r_k / \partial x_j) \mathcal{X}_j$ $k = 1, \dots, s$.

The main difficulty in deciding whether a generating system $\mathcal{V} = \{v_1, \dots, v_n\}$ is Nielsen equivalent to the system $\mathcal{U} = \{u_1, \dots, u_n\}$, with $u_i = \varphi(x_i)$, is the existence of different choices of lifts $y_1, \dots, y_n \in F(x_1, \dots, x_n)$ for v_1, \dots, v_n . If one can find a ring homomorphism $\rho: \mathbb{Z}G \rightarrow R$ in which $\rho(\varphi(\partial r_k / \partial x_j)) = 0$ for all $k = 1, \dots, s, j = 1, \dots, n$, this difficulty is resolved as $\rho(\varphi(\partial y_i / \partial x_j))$ is uniquely determined by \mathcal{U} and \mathcal{V} (Lemma 1.2). Consequently, if such a ρ exists, we will denote $\rho(\varphi(\partial y_i / \partial x_j))_{i,j}$ by $\rho(\partial \mathcal{V} / \partial \mathcal{U})$. Applying this to two systems of generators which are Nielsen equivalent, we have:

PROPOSITION 1.4. *Let $\langle x_1, \dots, x_n | r_1, \dots, r_s \rangle$ be a presentation for G which specifies an epimorphism $\varphi: F(x_1, \dots, x_n) \twoheadrightarrow G$ with $\varphi(x_i) = u_i$. Assume $\rho: \mathbb{Z}G \rightarrow R$ is a ring homomorphism satisfying $\rho(\varphi(\partial r_k / \partial x_j)) = 0$, for $k = 1, \dots, s, j = 1, \dots, n$. Let $\mathcal{V} = \{v_1, \dots, v_n\}$ be a generating system equivalent to $\mathcal{U} = \{u_1, \dots, u_n\}$. Then for any elements $y_1, \dots, y_n \in F(x_1, \dots, x_n)$ such that $\varphi(y_i) = v_i$, the matrix $\rho(\varphi(\partial y_i / \partial x_j))_{i,j} = \rho(\partial \mathcal{V} / \partial \mathcal{U})$ is a finite product of elementary R -matrices.*

Proof. As \mathcal{U} and \mathcal{V} are Nielsen equivalent there exists a basis $Y = \{y_1, \dots, y_n\}$ of $F_n = F(x_1, \dots, x_n)$ such that $\varphi(y_i) = v_i$. As discussed above, the matrix $\varphi(\partial y_i / \partial x_j)_{i,j}$ is a product of elementary $\mathbb{Z}G$ -matrices, hence $\rho(\varphi(\partial y_i / \partial x_j)_{i,j})$ is a product of elementary R -matrices. If $Y' = \{y'_1, \dots, y'_n\} \subset F(x_1, \dots, x_n)$ is any other lift of \mathcal{V} then, by Lemma 1.2, the difference matrix $(\varphi(\partial y_i / \partial x_j) - \varphi(\partial y'_i / \partial x_j))_{i,j}$ has all entries in $\ker \rho$. \square

For the rest of the paper G will denote a Fuchsian group with a fixed presentation:

$$\langle q_1, \dots, q_m, a_1, b_1, \dots, a_g, b_g | q_i^{\alpha_i} (i = 1, \dots, m), q_1, \dots, q_m \sum_{l=1}^g [a_l, b_l] \rangle, \quad m \geq 3 \text{ or } g \geq 1. \quad (1)$$

The group G is generated by any system $\{q_1^{\alpha_1}, \dots, q_{j-1}^{\alpha_{j-1}}, q_{j+1}^{\alpha_{j+1}}, \dots, q_m^{\alpha_m}, a_1, b_1, \dots, a_g, b_g\}$ with $v_i \in \mathbb{Z}$, $1 \leq v_i < \alpha_i/2$, $\text{g.c.d.}(v_i, \alpha_i) = 1$. For notational convenience we denote such a system by $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$. In particular Rosenberger proves, in [16], that if $g \geq 1$ and $m \geq 2$, then any system $\{x_1, \dots, x_{2g+m-1}\}$ of generators for G is Nielsen equivalent to some system $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$.

Any generating system $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$ determines an epimorphism $\varphi: F(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, a_1, b_1, \dots, a_g, b_g) \rightarrow G$ with $\varphi(x_i) = q_i^{\alpha_i}$, $\varphi(a_l) = a_l$, $\varphi(b_l) = b_l$. A presentation for G with respect to φ is given by:

$$\begin{aligned} \langle X | R \rangle \cong \langle x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, \\ a_1, b_1, \dots, a_g, b_g | x_i^{\alpha_i} (i = 1, \dots, j-1, j+1, \dots, m), \\ (x_{j+1}^{\mu_{j+1}}, \dots, x_m^{\mu_m}) \prod_{l=1}^g [a_l, b_l] x_1^{\mu_1} \dots x_{j-1}^{\mu_{j-1}} \rangle, \end{aligned} \quad (2)$$

where $\mu_i v_i \equiv \pm 1 \pmod{\alpha_i}$.

Every Fuchsian group G has a faithful representation ρ into $\text{PSL}_2(\mathbb{C})$. The image $\rho(G)$ acts as a group of fractional linear transformations on the unit disk model of the hyperbolic plane \mathbb{H}^2 . Each $\rho(q_i)$, $i = 1 \dots m$, is a rotation of angle $2\pi/\alpha_i$ about some point in \mathbb{H}^2 . The representation ρ lifts to a (faithful) representation $\rho: G \rightarrow \text{SL}_2(\mathbb{C})$ if and only if all exponents α_i are odd (see [8, 9]). The map ρ extends naturally to a ring homomorphism, also denoted by ρ , from $\mathbb{Z}G$ into the (2×2) -matrix ring $\mathbb{M}_2(\mathbb{C})$. We now claim:

LEMMA 1.5. *Let G be a Fuchsian group with a representation ρ into $\text{SL}_2(\mathbb{C})$ as above. Any presentation $\langle X | R \rangle$ of G , derived from a generating system $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$ as in (2), satisfies $\rho(\partial r / \partial x) = 0$ for each $x \in X$, $r \in R$.*

Proof. Note that any $r \in R$ is of the form w^n , $n > 1$, n odd. Hence

$$\partial r / \partial x = (1 + w + w^2 + \dots + w^{n-1}) \partial w / \partial x.$$

The element $\varphi(w)$ has finite order n and therefore is represented by an elliptic element of $\text{PSL}_2(\mathbb{C})$. Hence it can be conjugated in $\text{PSL}_2(\mathbb{C})$ to a matrix $\begin{pmatrix} e^{i\pi k/n} & 0 \\ 0 & e^{-i\pi k/n} \end{pmatrix}$, defined up to sign, with $(k, n) = 1$ and $1 \leq k \leq n$. It follows that if k is odd then, up to conjugation,

$$\rho(w) = \begin{pmatrix} -e^{i\pi k/n} & 0 \\ 0 & -e^{-i\pi k/n} \end{pmatrix}$$

in $SL_2(\mathbb{C})$, and if k is even

$$\rho(w) = \begin{pmatrix} e^{i\pi k/n} & 0 \\ 0 & e^{-i\pi k/n} \end{pmatrix},$$

as in both cases the other possible lift has order $2n$ in $SL_2(\mathbb{C})$. Consequently we have (up to a suitable conjugation):

$$\begin{aligned} \rho(\varphi((1+w+w^2+\dots+w^{n-1}))) &= \begin{bmatrix} \sum_{j=0}^{n-1} (-1)^{jk} e^{i\pi jk/n} & 0 \\ 0 & \sum_{j=0}^{n-1} (-1)^{jk} e^{-i\pi jk/n} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

since $(-1)^k e^{i\pi k/n}$ is a primitive n th root of unity. \square

An $(n \times n)$ -matrix with entries in $\mathbb{M}_2(\mathbb{C})$ can be thought of as a $(2n \times 2n)$ -matrix with entries in \mathbb{C} . This "deleting the bracket" map preserves matrix multiplication. Hence for any matrix $A \in \mathbb{M}_n(\mathbb{Z}G)$ the value $\det \rho(A) \in \mathbb{C}$ is well defined.

LEMMA 1.6. *Let A be an elementary matrix over $\mathbb{Z}G$. Then $\det \rho(A) = 1$.*

Proof. If A has a unique non-zero off diagonal element and 1's on the diagonal the claim is obvious. If A has an element $\pm g \in G$ on the diagonal the claim follows from the fact that $\rho(\pm g) \in SL_2(\mathbb{C})$. \square

Combining Proposition 1.4, Lemma 1.5 and Lemma 1.6 we obtain:

PROPOSITION 1.7. *Let G be a Fuchsian group with no elements of even order and $\rho: G \rightarrow SL_2(\mathbb{C})$ be a faithful representation, which induces an action on the unit disk (by hyperbolic isometries).*

*Let $\mathcal{U} = \mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$ and $\mathcal{V} = \{v_1, \dots, v_{2g+m-1}\}$ be two Nielsen equivalent systems of generators for G . Then $\det \rho(\partial \mathcal{V} / \partial \mathcal{U}) = 1$.* \square

COROLLARY 1.8. *Let G , ρ and $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$ be as in Proposition 1.7. Then for any generating system $\mathcal{V} = \{v_1, \dots, v_{2g+m-1}\}$ the number $\det \rho(\partial \mathcal{V} / \partial \mathcal{U})$ does not depend on \mathcal{U} and \mathcal{V} but only on their Nielsen equivalence classes.* \square

We are now ready for the proof of Theorem 1 (stated in §0.).

Proof of Theorem 1. If the conditions of case (i) are satisfied then \mathcal{U} and \mathcal{V} are trivially Nielsen equivalent. In case (ii) consider the free group

$$F_{2g+m-1} = F(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_n, a_1, b_1, \dots, a_g, b_g)$$

and the map $\varphi: F_{2g+m-1} \rightarrow G$ given by $\varphi(x_i) = q_i^{\mu_i}$, $\varphi(a_i) = a_i$, $\varphi(b_i) = b_i$. If $\mu_i \in \mathbb{Z}$ is such that $\mu_i v_i \equiv \pm 1 \pmod{\alpha_i}$ then the elements $x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{k-1}, x_{k+1}, \dots, x_n, a_1, b_1, \dots, a_g, b_g$ together with the element $(x_{j+1}^{\mu_{j+1}} \dots x_{k-1}^{\mu_{k-1}}, x_k x_{k+1}^{\mu_{k+1}} \dots x_m^{\mu_m} \prod_{l=1}^g [a_l, b_l] x_1^{\mu_1} \dots x_{j-1}^{\mu_{j-1}})^{-1}$ clearly constitute a base for F_{2g+m-1} which is mapped, by φ , onto $\mathcal{V}'(v'_1, \dots, v'_{k-1}, *, v'_{k+1}, \dots, v'_m)$.

For the converse direction we distinguish again the two cases:

(i) Let $j = k$. In G we have $v_i = q_i^{v_i} = (q_i^{v_i})^{\eta_i} = u_i^{\eta_i}$ for some $1 \leq \eta_i \leq \alpha_i$. Hence the matrix $\rho(\partial \mathcal{V} / \partial \mathcal{U})$ is a block diagonal matrix. Each block is a (2×2) -matrix equal to either

$$\rho(\partial v_i / \partial u_i) = \rho(\partial u_i^{\eta_i} / \partial u_i) = \rho(1 + u_i + u_i^2 + \dots + u_i^{\eta_i-1})$$

$$= \rho(1 + q_i^{v_i} + (q_i^{v_i})^2 + \dots + (q_i^{v_i})^{\eta_i-1}), \text{ or to}$$

$$\rho(\partial a_i / \partial a_i) = \rho(\partial b_i / \partial b_i) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As $\rho(q_i)$ is a rotation of angle $2\pi/\alpha_i$ of \mathbb{H}^2 the matrix $\rho(q_i) \in \mathrm{SL}_2(\mathbb{C})$ can be conjugated in $\mathrm{SL}_2(\mathbb{C})$ to $\begin{bmatrix} -e^{i\pi/\alpha_i} & 0 \\ 0 & -e^{-i\pi/\alpha_i} \end{bmatrix}$. Hence $\rho(1 + q_i^{v_i} + (q_i^{v_i})^2 + \dots + (q_i^{v_i})^{\eta_i-1})$ is conjugate to

$$\begin{bmatrix} \sum_{l=0}^{\eta_i-1} (-1)^{lv_i} e^{i\pi lv_i/\alpha_i} & 0 \\ 0 & \sum_{l=0}^{\eta_i-1} (-1)^{lv_i} e^{-i\pi lv_i/\alpha_i} \end{bmatrix}$$

Denote $(-1)^{lv_i} e^{i\pi lv_i/\alpha_i}$ by ξ_l . It follows that

$$\det \rho(\partial \mathcal{V} / \partial \mathcal{U}) = \prod_{\substack{l=1 \\ i \neq k}}^m \det \begin{bmatrix} \sum_{l=0}^{\eta_i-1} \xi_l^{lv_i} & 0 \\ 0 & \sum_{l=0}^{\eta_i-1} \bar{\xi}_l^{lv_i} \end{bmatrix} = \prod_{\substack{l=1 \\ i \neq k}}^m \left| \frac{1 - \xi_l^{\eta_i v_i}}{1 - \xi_l^{v_i}} \right|^2 = \prod_{\substack{l=1 \\ i \neq k}}^m \left| \frac{1 - \xi_l^{v_i}}{1 - \xi_l^{v_i}} \right|^2$$

(ii) Let $j \neq k$. For $i \neq j, k$ we get again $v_i = q_i^{v_i} = (q_i^{v_i})^{\eta_i} = u_i^{\eta_i}$. We reorder the generating set \mathcal{V} by relabelling the generator $q_j^{v_j}$ as v_k , hence:

$$v_k = q_j^{v_j} = (u_{j+1}^{\mu_{j+1}} \dots u_m^{\mu_m} \prod_{l=1}^g [a_l, b_l] u_1^{\mu_1} \dots u_{j-1}^{\mu_{j-1}})^{-v_j'}$$

Now the matrix $\rho(\partial \mathcal{V} / \partial \mathcal{U})$ is a block diagonal matrix with (2×2) -blocks, except for (2×2) -matrix entries:

$$\rho\left(\frac{\partial v_k}{\partial u_1}\right), \dots, \rho\left(\frac{\partial v_k}{\partial u_{j-1}}\right), \rho\left(\frac{\partial v_k}{\partial u_{j+1}}\right), \dots, \rho\left(\frac{\partial v_k}{\partial u_m}\right),$$

$$\rho\left(\frac{\partial v_k}{\partial a_1}\right), \rho\left(\frac{\partial v_k}{\partial b_1}\right), \dots, \rho\left(\frac{\partial v_k}{\partial a_g}\right), \rho\left(\frac{\partial v_k}{\partial b_g}\right)$$

in the k th row. Hence $\det \rho(\partial \mathcal{V} / \partial \mathcal{U})$ is equal to the product of determinants of the (2×2) -matrices M_i in the diagonal of $\rho(\partial \mathcal{V} / \partial \mathcal{U})$. For $i \neq k$ we obtain as in case (i) the equalities $\det M_i = \left| \frac{1 - \xi_i^{v_i}}{1 - \xi_i^{v_i}} \right|^2$. For $i = k$ we use the product rule for Fox derivatives to compute:

$$M_k = \rho(\partial v_k / \partial u_k) = \rho(\partial q_j^{v_j} / \partial u_k) = \rho(1 + q_j + \dots + (q_j)^{v_j-1}) \rho(\partial q_j / \partial u_k).$$

Since the representation ρ sends group elements with either positive or negative sign to matrices with determinant equal to ± 1 , we get

$$\det \rho(\partial q_j / \partial u_k) = \det \rho(1 + q_k^{v_k} + (q_k^{v_k})^2 + \dots + (q_k^{v_k})^{\mu_k-1}) = \left| \frac{1 - \xi_k}{1 - \xi_k^{v_k}} \right|^2,$$

and hence we can compute,

$$\det M_k = \left| \frac{1 - \xi_j^{\nu_j}}{1 - \xi_j} \right|^2 \left| \frac{1 - \xi_k}{1 - \xi_k^{\nu_k}} \right|^2$$

As a conclusion we obtain in case (ii):

$$\det \rho(\partial \mathcal{V} / \partial \mathcal{U}) = \left| \frac{1 - \xi_j^{\nu_j}}{1 - \xi_j} \right|^2 \left| \frac{1 - \xi_k}{1 - \xi_k^{\nu_k}} \right|^2 \prod_{i=1, i \neq j, k}^m \left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i} \right|^2 \quad (3)$$

The statement of Theorem 1 is now an immediate consequence of the following Lemma and Proposition 1.7. \square

LEMMA 1.9. *Let $\alpha_1, \dots, \alpha_N$ be a set of pairwise relatively prime integers, and let $v_i, v'_i \in \mathbb{Z}$ be relatively prime to α_i with $1 \leq v_i, v'_i < \alpha_i/2$, for all $i = 1, \dots, N$. Then the equation:*

$$\prod_{i=1}^N \left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right|^2 = 1, \quad \xi_i = -e^{\pi i / \alpha_i}, \text{ implies } v_i = v'_i \text{ for all } i = 1, \dots, N. \quad (4)$$

Proof. Let $\eta_i \in \mathbb{Z}$ such that $\nu'_i \equiv \eta_i \nu_i \pmod{\alpha_i}$. Then

$$\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right|^2 = \left(\frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right) \left(\frac{1 - \bar{\xi}_i^{\nu_i}}{1 - \bar{\xi}_i^{\nu'_i}} \right) = (1 + \xi_i^{\nu_i} + \dots + \xi_i^{(\eta_i-1)\nu_i}) (1 + \bar{\xi}_i^{-\nu_i} + \dots + \bar{\xi}_i^{-(\eta_i-1)\nu_i}).$$

This shows that any expression $\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right|^2$ is contained in the ring $\mathbb{Z}(\xi_i) = \mathbb{Z}(e^{2\pi i / \alpha_i})$. This ring coincides with the ring of algebraic integers of the cyclotomic field $\mathbb{Q}(e^{2\pi i / \alpha_i})$ (see [18], p. 11).

Hence for each $i = 1, \dots, N$ the factor $\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right|^2$ is an algebraic integer. As its inverse is of the same form, and hence also an algebraic integer, each $\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right|^2$ is a unit in $\mathbb{Z}(e^{2\pi i / \alpha_i})$.

For any two relatively prime $\alpha_i, \alpha_l \in \mathbb{Z}$ the cyclotomic fields $\mathbb{Q}(\xi_i) = \mathbb{Q}(e^{2\pi i / \alpha_i})$ and $\mathbb{Q}(\xi_l) = \mathbb{Q}(e^{2\pi i / \alpha_l})$ intersect exactly in \mathbb{Q} (see [18], p. 11). Define $d(k) = \text{l.c.m.}\{\alpha_l \mid l \neq k\}$. We get from equation (4) that

$$\left| \frac{1 - \xi_k^{\nu_k}}{1 - \xi_k^{\nu'_k}} \right|^2 = \prod_{\substack{l=1 \\ l \neq k}}^N \left| \frac{1 - \xi_l^{\nu_l}}{1 - \xi_l^{\nu'_l}} \right|^2 \in \mathbb{Q}(\xi_k) \cap \mathbb{Q}(\xi_{d(k)}) = \mathbb{Q}.$$

The only mutually inverse algebraic integers in \mathbb{Q} are 1 and -1 , hence we obtain

$$\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i^{\nu'_i}} \right| = 1 \quad \text{for all } i = 1, \dots, N. \quad \text{Furthermore one computes:}$$

$$|1 - \xi_i^{\nu_i}| = |1 - (-1)^{\nu_i} e^{i\pi \nu_i / \alpha_i}| =$$

$$|e^{i\pi \nu_i / 2\alpha_i}| |e^{-i\pi \nu_i / 2\alpha_i} - (-1)^{\nu_i} e^{i\pi \nu_i / 2\alpha_i}| = \begin{cases} 2|\sin(\pi \nu_i / 2\alpha_i)| & \text{if } \nu_i \text{ is even} \\ 2|\cos(\pi \nu_i / 2\alpha_i)| & \text{if } \nu_i \text{ is odd.} \end{cases}$$

From our normalization condition $1 \leq v_i, v'_i < \alpha_i/2$ we deduce $0 \leq \pi \nu_i / 2\alpha_i < \pi/4$, for which the values of $\sin(x)$ and $\cos(x)$ are disjoint. Hence $|1 - \xi_i^{\nu_i}| = |1 - \xi_i^{\nu'_i}|$ implies $\nu_i = \nu'_i$. \square

The proof of Theorem 1 gives the following additional fact, which we find quite surprising:

COROLLARY 1.10. *For generating systems $\mathcal{U}(v_1, \dots, v_{j-1}, *, v_{j+1}, \dots, v_m)$ and $\mathcal{V}(v'_1, \dots, v'_{k-1}, *, v'_{k+1}, \dots, v'_m)$ of G and any representation $\rho: G \rightarrow \mathbb{M}_2(\mathbb{C})$ as in Proposition 1.7 the value of $\det \rho(\partial \mathcal{V} / \partial \mathcal{U})$ is independent of the particular choice of ρ . It is always*

a positive real algebraic number (cyclotomic unit), which is given by the formula:

$$\mathcal{V}(\mathcal{Y}/\mathcal{U}) = \det \rho(\partial \mathcal{Y} / \partial \mathcal{U}) = \prod_{i=1}^m \left| \frac{1 - (-e^{i\pi \alpha_i})^{v_i}}{1 - (-e^{-i\pi \alpha_i})^{v_i}} \right|^2$$

with the convention that $v_j = v_k = 1$. □

Because of the chain rule for Fox derivatives one can associate with each of the Nielsen equivalence classes (represented by one of the given generating systems \mathcal{U}) a value $\mathcal{V}(\mathcal{U}) \in \mathbb{R}^+$ which satisfies:

$$\mathcal{V}(\mathcal{Y}/\mathcal{U}) = \mathcal{V}(\mathcal{Y}) / \mathcal{V}(\mathcal{U}).$$

The function $\mathcal{V}(\mathcal{U})$ is well defined if we normalize as follows:

$$\mathcal{V}(\mathcal{U}(1, \dots, 1, *, 1, \dots, 1)) = 1.$$

It is easy to see then that all other $\mathcal{V}(\mathcal{Y})$ lie in the interval $[0, 1]$.

Remark 1.11. The restrictions on the exponents α_i that appear in Theorem 1 are a consequence of the proof; the analogous statement under general conditions will be shown in [10]. One sees immediately that the invariant $\mathcal{V}(\mathcal{Y}/\mathcal{U})$ is defined for Fuchsian groups G as in (1) also in the case of non-relatively prime exponents α_i (see Proposition 1.7 and Corollary 1.8). It can be used to show that, whenever all α_i are odd and at least one of them is bigger than 3, there is more than one Nielsen equivalence class of generating systems with $2g + m - 1$ elements. If G has no elements of order 2^n a more general result follows from quotienting G with possibly even α_i to a Fuchsian group G' with odd exponents α'_i (where α'_i divides α_i).

§2. THE HEEGAARD STRUCTURE

In this section M will always denote an orientable Seifert fibered space over an orientable surface. Each such manifold is characterized by a system of invariants $\{g, e, (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ where g is the genus of the orbit space, e measures the Euler class of the fibration and (α_i, β_i) are the normalized Seifert invariants of the exceptional fibers, with $\text{g.c.d.}(\alpha_i, \beta_i) = 1$, $1 \leq \beta_i \leq \alpha_i$ (see [17]). We will assume that $m \geq 2$ and if $m = 2$ then $g > 0$. A Heegaard splitting of genus g for a manifold M is standardly defined as a decomposition $M = H_1 \cup H_2$, where $H_1 \cap H_2 = \partial H_1 = \partial H_2$ and H_i is a genus g handlebody. We first describe “vertical” Heegaard splittings of M of genus $2g + m - 1$. This generalizes Proposition 1.3 of [2], from which we take the notation, and constructions used previously in [3], [4] and [15]. These “vertical” Heegaard splittings are of minimal genus for the manifolds considered in Theorem 2, as the α_i ’s are all odd.

Let B denote the orbit space of M . Let $X_1, \dots, X_m \subset M$ denote the exceptional fibers and x_1, \dots, x_m be their images in B . Choose a system of disjoint disks D_1, \dots, D_m on B , centered at x_1, \dots, x_m respectively. For an arbitrary choice of $i_0 \in \{1, \dots, m\}$ we may choose a base point x_0 on the boundary of D_{i_0} and proceed as follows: As indicated in Fig. 1, we specify the following simple curves on B which are all based at x_0 and otherwise disjoint. For $i = 1, \dots, m$ we denote by σ_i an arc joining x_i to x_0 , and by q_i a closed curve which winds once around the point x_i (without intersecting D_i). For $l = 1, \dots, g$ let a_l, b_l be a pair of dual closed curves, such that B , when cut open along $\{a_1, b_1, \dots, a_g, b_g\}$, becomes a disk. Denote the preimage of D_i in M by V_i . The V_i ’s are solid tori, which are regular neighbor-

hoods of the exceptional fibers. As M and B are orientable and $M - \bigcup_{i=1}^m \dot{V}_i$ is a trivial S^1 fibration over $B - \bigcup_{i=1}^m \dot{D}_i$, we can lift $B - \bigcup_{i=1}^m \dot{D}_i$ with all specified curves, into $M - \bigcup_{i=1}^m \dot{V}_i \subset M$.

The curves q_i on B will be called "horizontal", whereas the exceptional fibers X_i will be called "vertical". The Heegaard splittings Σ of M under consideration will be constructed by choosing regular neighborhoods of $a_1, b_1, \dots, a_g, b_g$, together with a regular neighborhood of a combination of $m-1$ vertical and horizontal curves. Each index $i \in \{1, \dots, m\}$ will contribute a vertical curve in exactly one of the two handlebodies of the splitting Σ , and this partition determines the Heegaard splitting.

More precisely: Choose some $i_0 \in \{1, \dots, m\}$ and consider the above defined data. Let $i_1, i_2, \dots, i_j \in \{1, \dots, m\} - \{i_0\}$ be a collection of distinct indices, with $0 \leq j \leq m-2$. Let $\Gamma(i_0, i_1, \dots, i_j)$ be the graph in M which is the union of $a_1, b_1, \dots, a_g, b_g, \sigma_{i_0}, \sigma_{i_1}, \dots, \sigma_{i_j}, X_{i_0}, X_{i_1}, \dots, X_{i_j}$ and any choice of $m-j-2$ horizontal curves q_t , $t \in \{1, \dots, m\} - \{i_0, i_1, \dots, i_j\}$, all based at the base point x_0 in D_{i_0} . The regular neighbourhood $N(\Gamma(i_0, i_1, \dots, i_j))$ is clearly a handlebody of genus $2g + m - 1$. In order to see that the closure of the complement of $N(\Gamma(i_0, i_1, \dots, i_j))$, in M , is also a genus $2g + m - 1$ handlebody, we consider the $2g$ thickened segments on $a_1, b_1, \dots, a_g, b_g$, the j thickened segments on $\sigma_{i_1}, \dots, \sigma_{i_j}$ and the $m-j-2$ thickened segments on the chosen q_t 's, all with endpoints in $N(\Gamma(i_0, i_1, \dots, i_j))$, as indicated in Fig. 1.

The preimage, in M , of each segment is an annulus, and the intersection of this annulus with $M - \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$ is a disk. Cutting along these disks we obtain $m-j-1$ disjoint solid tori, which are regular neighbourhoods of the corresponding $m-j-1$ exceptional fibers X_k , $k \neq i_0, i_1, \dots, i_j$. Conversely, gluing the tori back together along the corresponding disks, as well as gluing the remaining pairs of disks (coming from the σ_i 's and the a_i 's and b_i 's), exhibits the complement of $M - \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$ as a genus $2g + m - 1$ handlebody.

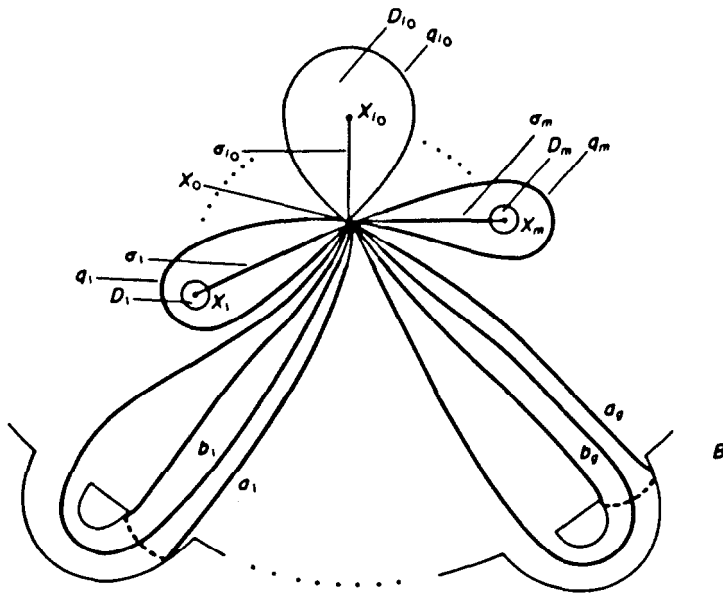


Fig. 1.

For the following remarks we need to point out that cores of the handles for this complementary handlebody are obtained as follows: Choose a second embedding of $B = \bigcup_{i=1}^m D_i$ in $M = \bigcup_{i=1}^m V_i = \left\{ B = \bigcup_{i=1}^m D_i \right\} \times S^1$, which is a push off in the fiber direction of the original embedding. Denote it by $B' = \bigcup_{i=1}^m D'_i$. Choose a point x'_0 on $\partial D'_{i_0}$, where i_0 is the well defined index which did not occur among the i_0, i_1, \dots, i_j and the l_1, \dots, l_{m-j-2} . On $B' = \bigcup_{i=1}^m D'_i$ consider the curves $\sigma'_i, q'_i, a'_j, b'_j$ as defined earlier in the section, but now with x'_0 on $\partial D'_i$ as base point. From the above described decomposition it is immediate that the curves:

$$a'_1, b'_1, \dots, a'_g, b'_g, \text{ and } \sigma'_{i_0}, \sigma'_{i_1}, \dots, \sigma'_{i_{m-j-2}}, X_{i_0}, X_{i_1}, \dots, X_{i_{m-j-2}}, q'_{i_1}, \dots, q'_{i_j}$$

constitute a core for the handlebody $M = \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$.

Remark 2.1. The isotopy class of any handlebody $N(\Gamma(i_0, i_1, \dots, i_j))$ (and correspondingly of $M = \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$) is independent of the choice of i_0 among i_0, i_1, \dots, i_j , the indices of the vertical curves of the handlebody. It is easy to see that M has an isotopy “shrinking” D_{i_0} and “blowing up” some other D_i , $i \in \{i_0, i_1, \dots, i_j\}$, so x_0 comes to lie on ∂D_i instead of ∂D_{i_0} . The effect of this isotopy on $M = \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$ is to replace the horizontal 1-handle corresponding to q'_i by a horizontal 1-handle corresponding to q'_{i_0} .

Remark 2.2. The isotopy class of the Heegaard splitting Σ of M given by $H_1 = N(\Gamma(i_0, i_1, \dots, i_j))$ and $H_2 = M = \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$ is independent of (a) the choice of arcs σ_i, a_i, b_i, q_i in B , and (b) the choice of the embedding of $B = \bigcup_{i=1}^m D_i$ in M . For

(a) notice that (2-dimensional) regular neighbourhoods in $B = \bigcup_{i=1}^m D_i$ of different systems of curves are isotopic in B . For (b) notice that any handlebody $N(\Gamma(i_0, i_1, \dots, i_j))$ contains a vertical handle with index i_0 . Sliding an endpoint p of the thickened segment on σ_i, a_i, b_i, q_i along the regular fiber which is the preimage of p , contained in $\partial N(\Gamma(i_0, i_1, \dots, i_j))$, describes an isotopy of M . For a different embedding of $B = \bigcup_{i=1}^m D_i$, a suitable sequence of such isotopies will take $N(\Gamma(i_0, i_1, \dots, i_j))$ to a regular neighborhood of the corresponding curves on $B = \bigcup_{i=1}^m D_i$.

Remark 2.3. The handlebody $N(\Gamma(i_0, i_1, \dots, i_j))$ is isotopic to the complement of the handlebody $N(\Gamma(l_0, l_1, \dots, l_{m-j-2}))$, determined by the complementary set of indices $\{l_0, l_1, \dots, l_{m-j-2}\} = \{1, \dots, m\} - \{i_0, i_1, \dots, i_j\}$ for the vertical handles, and indices $\{i_1, \dots, i_j\}$ for horizontal ones. Note that the isotopy between the handlebodies takes the curve a_i to b'_i and b_i to a'_i . Also note that for all the Heegaard splittings described so far the curves a_i, b_i can be taken fixed except for in a neighbourhood of x_0 .

Remark 2.4. The construction of the Heegaard splittings and Remarks 2.1, 2.2 and 2.3 show that the Heegaard splittings, up to isotopy, depend only on the distribution of the vertical curves X_i in the two handlebodies. Taking into account all possibilities we get, up to

isotopy, $\sum_{k=1}^{m-1} \binom{m}{k} / 2 = 2^{m-1} - 1$ genus $2g + m - 1$ Heegaard splittings, which so far are candidates to be non-isotopic.

Definition 2.5. A Heegaard splitting Σ for M with $H_1 = N(\Gamma(i_0, i_1, \dots, i_j))$ and $H_2 = M - \text{int } N(\Gamma(i_0, i_1, \dots, i_j)) = N(\Gamma(l_0, l_1, \dots, l_{m-j-2}))$ will be denoted by $\Sigma(i_0, i_1, \dots, i_j)$ or by $\Sigma(l_0, l_1, \dots, l_{m-j-2})$ (for $\{i_0, i_1, \dots, i_j, l_0, l_1, \dots, l_{m-j-2}\} = \{1, \dots, m\}$). All such splittings will be called *vertical*.

The remainder of this section will be devoted to determining which of these vertical $2^{m-1} - 1$ splittings are isotopic (or homeomorphic), and which are not. From the above described cores of the handlebodies of a vertical Heegaard splitting we immediately get:

LEMMA 2.6. Let $\Sigma(i_0, i_1, \dots, i_j)$ be a vertical Heegaard splitting of M . Denote by Y_i the curve $\sigma_i X_i \sigma_i^{-1}$, then:

- (i) The fundamental group of the handlebody $H_1 = N(\Gamma(i_0, i_1, \dots, i_j))$ is generated by $a_1, b_1, \dots, a_g, b_g, Y_{i_0}, Y_{i_1}, \dots, Y_{i_j}, q_{l_1}, \dots, q_{l_{m-j-2}}$ for any $m - j - 2$ elements $l_1, \dots, l_{m-j-2} \in \{1, \dots, m\} - \{i_0, i_1, \dots, i_j\}$.
- (ii) The fundamental group of the handlebody $H_2 = M - \text{int } N(\Gamma(i_0, i_1, \dots, i_j))$ is generated by $a_1, b_1, \dots, a_g, b_g, Y_{l_0}, Y_{l_1}, \dots, Y_{l_{m-j-2}}, q_{i_1}^2, \dots, q_{i_j}^2$, where l_0 is the unique element in $\{1, \dots, m\} - \{i_0, i_1, \dots, i_j, l_1, \dots, l_{m-j-2}\}$. \square

The suitably oriented curves $q_1, \dots, q_m, a_1, b_1, \dots, a_g, b_g$ together with a regular fiber h constitute another generating system for $\pi_1(M)$. A presentation for $\pi_1(M)$ with respect to this generating system is given by: $\pi_1(M) \cong \langle q_1, \dots, q_m, a_1, b_1, \dots, a_g, b_g, h \mid [h, q_i], [h, a_i], [h, b_i], q_i^{\beta_i} h^{\beta_i}, h^{\alpha_i} q_1, \dots, q_m \prod_{l=1}^g [a_l, b_l] \rangle$, where $i = 1, \dots, m$ and $l = 1, \dots, g$ (see [17]). The fiber h generates the center of $\pi_1(M)$. The quotient group $\pi_1(M)/\langle h \rangle$ is a Fuchsian group G as in §1, with presentation $G \cong \langle q_1, \dots, q_m, a_1, b_1, \dots, a_g, b_g \mid q_i^{\beta_i}, q_1, \dots, q_m \prod_{l=1}^g [a_l, b_l] \rangle$. Consider the regular neighbourhood V_i of an exceptional fiber X_i in M . If μ_i, λ_i are a meridian-longitude pair for $H_1(\partial V_i)$ and h is the image in $H_1(\partial V_i)$ of a regular fiber, then

$$h = \mu_i^{\beta_i} \lambda_i^{\beta_i}, q_i = \mu_i^{-\delta_i} \lambda_i^{-\beta_i} \text{ and } \lambda_i = q_i^{-\gamma_i} h^{-\delta_i}, \mu_i = q_i^{\alpha_i} h^{\beta_i}, \quad (5)$$

where $\beta_i \gamma_i - \alpha_i \delta_i = 1$. We can also assume that $1 \leq \gamma_i < \alpha_i$ (see [17]). In M the longitude λ_i is homotopic to X_i and μ_i is null homotopic. Hence in $\pi_1(M)/\langle h \rangle$ we obtain $X_i = \lambda_i = q_i^{-\gamma_i}$ and $X_i^{\alpha_i} = q_i^{\alpha_i \gamma_i} = 1$. We summarize:

LEMMA 2.7. Given a Seifert fibered space M and a Heegaard splitting $\Sigma(i_0, i_1, \dots, i_j)$, then:

- (i) The set of "natural" generators, given in Lemma 2.6, for $\pi_1(H_1)$ and hence for $\pi_1(M)$ maps to the generators:

$$\{q_{i_0}^{-\gamma_{i_0}}, q_{i_1}^{-\gamma_{i_1}}, \dots, q_{i_j}^{-\gamma_{i_j}}, q_{l_1}, \dots, q_{l_{m-j-2}}, a_1, b_1, \dots, a_g, b_g\} \text{ in } \pi_1(M)/\langle h \rangle.$$

- (ii) The set of "natural" generators, given in Lemma 2.6, for $\pi_1(H_2)$ and hence for $\pi_1(M)$ maps to the generators:

$$\{q_{l_0}^{-\gamma_{l_0}}, q_{l_1}^{-\gamma_{l_1}}, \dots, q_{l_{m-j-2}}^{-\gamma_{l_{m-j-2}}}, q_{i_1}, \dots, q_{i_j}, a_1, b_1, \dots, a_g, b_g\} \text{ in } \pi_1(M)/\langle h \rangle.$$

(iii) The set of “natural” generators of any vertical Heegaard splitting isotopic to $\Sigma(i_0, i_1, \dots, i_j)$, as in Remarks 2.1–2.3, maps to generators Nielsen equivalent to generators as in (i) or (ii). \square

If the Heegaard splitting Σ of M is vertical, $\Sigma = \Sigma(i_0, i_1, \dots, i_j)$, then Lemma 2.7 shows that the Nielsen equivalence classes in G , determined by H_1 and H_2 , are represented by systems $\mathcal{U}(v_1, \dots, v_{r-1}, *, v_{r+1}, \dots, v_m)$ and $\mathcal{U}(v'_1, \dots, v'_{r'-1}, *, v'_{r'+1}, \dots, v'_m)$ respectively, as introduced in §1. More precisely, the “variables” v_i and v'_k of the systems are given by $v_{i_s} = \gamma_{i_s}$ for $s = 0, \dots, j$ and $v'_{i_t} = \gamma_{i_t}$ for $t = 0, \dots, m - j - 2$, and $v_i = v'_k = 1$ otherwise. The empty spot $*$ may be freely interchanged with any one of the last exponents (by Remark 2.1). This fact does not change the Nielsen equivalence class of the generating system (see Theorem 1.) We are ready to prove:

THEOREM 2.8. *Let M be a Seifert fibered space with Seifert invariants $\{g, e; (\alpha_1, \beta_1), \dots, (\alpha_m, \beta_m)\}$ satisfying $m = 2$ and $g > 0$ or $m \geq 3$ and the invariants α_i all odd and pairwise relatively prime. Let $\beta = \{i \mid \beta_i \neq \pm 1 \bmod \alpha_i; i = 1, \dots, m\}$. Then the two vertical Heegaard splittings $\Sigma(i_0, i_1, \dots, i_j)$ and $\Sigma(k_0, k_1, \dots, k_n)$ are isotopic if and only if the maximal subsets of β contained in $\{i_0, i_1, \dots, i_j\}$ and $\{k_0, k_1, \dots, k_n\}$ are either equal or complementary in β to each other.*

Proof. If $\Sigma(i_0, i_1, \dots, i_j)$ and $\Sigma(k_0, k_1, \dots, k_n)$ are isotopic, then the generating system of G corresponding to the handlebody H_1 , given by Lemma 2.7, must be Nielsen equivalent to the generating system corresponding to either H'_1 or H'_2 . The condition on the indices i_s and k_t , stated in the Theorem 2.8 is an immediate consequence of Theorem 1.

Conversely, assume that $\{i_0, i_1, \dots, i_j\} \cap \beta$ is equal to $\{k_0, k_1, \dots, k_n\} \cap \beta$. (If they are complementary we can rename $\Sigma(k_0, k_1, \dots, k_n)$ according to Definition 2.5.) It suffices to show that if $\beta_{i_s} = \pm 1 \bmod \alpha_{i_s}$ then $\Sigma(i_0, i_1, \dots, i_j)$ is isotopic to $\Sigma(i_0, i_1, \dots, i_{j-1})$. From (5), before Lemma 2.7, for each index i one obtains $q_i = \mu_i^{-\delta_i} \lambda_i^{-\beta_i}$. Hence the cross curve q_{i_j} is isotopic in M to the singular fiber $X_{i_j}^{\pm 1} = \lambda_{i_j}^{\pm 1}$ by an isotopy that pushes it across an annulus appropriately embedded in a fibered torus neighbourhood of X_{i_j} . This isotopy, extended to a regular neighborhood of the curves is the desired isotopy between $\Sigma(i_0, i_1, \dots, i_j)$ and $\Sigma(i_0, i_1, \dots, i_{j-1})$. \square

Theorem 2.8 also allows us to determine when two Heegaard splittings are isotopic by an isotopy preserving the order of the handlebodies. One can also show that M admits an isotopy exchanging the handlebodies if and only if for all $i = 1, \dots, m$ we have $\beta_i = \pm 1 \bmod \alpha_i$.

The question of deciding when two Heegaard splittings are homeomorphic is a more subtle one. A given homeomorphism $h: M \rightarrow M$ might induce an automorphism $h_*: \pi_1(M) \rightarrow \pi_1(M)$ which is not an inner automorphism. Now, in order to show that the two Heegaard splittings Σ_1 and Σ_2 are not homeomorphic, one needs to show that the generating system determined by H_1 of Σ_1 and the system obtained from the generators of say H'_1 of Σ_2 , after applying h_* , are not Nielsen equivalent. This leads to the rather delicate question of how $\text{Out}(\pi_1(M))$ acts on the Nielsen equivalent classes of $\pi_1(M)$. In [11] this question has been answered under more general conditions. As we will see now, for odd and pairwise relatively prime exponents α_i the problem can be solved using the invariants introduced in §1.

Definition 2.8. For any vertical Heegaard splitting Σ of M , let \mathcal{U} and \mathcal{V} denote generating systems of $G = \pi_1(M)/\langle h \rangle$ which correspond to H_1 and H_2 , respectively. We

define:

$$\mathcal{N}(\Sigma) = \min \{ \mathcal{N}(\mathcal{Y}/\mathcal{U}), \mathcal{N}(\mathcal{U}/\mathcal{Y}) \}$$

Because of the chain rule for Fox derivatives we have: $\mathcal{N}(\mathcal{Y}/\mathcal{U}) \cdot \mathcal{N}(\mathcal{U}/\mathcal{Y}) = \mathcal{N}(\mathcal{Y}/\mathcal{Y}) = 1$. Hence $0 < \mathcal{N}(\Sigma) \leq 1$. Since $\mathcal{N}(\mathcal{U}/\mathcal{Y})$ does not change under Nielsen transformation on \mathcal{U} or \mathcal{Y} , $\mathcal{N}(\Sigma)$ is a well defined invariant of the isotopy class of Σ . Recall that $\mathcal{N}(\mathcal{U}/\mathcal{Y}) = \det \rho(\partial \mathcal{U} / \partial \mathcal{Y})$ was computed from a faithful geometric (i.e. induced by a homeomorphism of the orbit space) representation ρ of G in $SL_2(\mathbb{C})$ and turned out to be independent of the particular choice of such a ρ (Corollary 1.10). By a Theorem of Zieschang, ([19] Theorem 5.8.3), any homeomorphism of M will induce an automorphism φ of G which is geometric. It follows that any such an automorphism φ of G will keep $\mathcal{N}(\mathcal{U}/\mathcal{Y}) = \mathcal{N}(\Sigma)$ invariant. In other words, $\mathcal{N}(\Sigma)$ is an invariant of the homeomorphism class of Σ . The fact that this invariant is complete is the content of Theorem 2, which we are now ready to prove.

Proof of Theorem 2. Let $\Sigma(i_0, i_1, \dots, i_j)$ and $\Sigma(k_0, k_1, \dots, k_n)$ be two vertical Heegaard splittings of M . If the splittings $\Sigma(i_0, i_1, \dots, i_j)$ and $\Sigma(k_0, k_1, \dots, k_n)$ are homeomorphic then by the remark above the invariants $\mathcal{N}(\Sigma(i_0, i_1, \dots, i_j))$ and $\mathcal{N}(\Sigma(k_0, k_1, \dots, k_n))$ are equal.

Suppose that $\mathcal{N}(\Sigma(i_0, i_1, \dots, i_j))$ and $\mathcal{N}(\Sigma(k_0, k_1, \dots, k_n))$ are equal. As all the α_i are odd and pairwise relatively prime, the equation of the invariants implies, by Lemma 1.9, that for each α_i the factor $\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i} \right|^2$ in $\mathcal{N}(\Sigma(i_0, i_1, \dots, i_j))$ is equal, up to inversion, to the factor $\left| \frac{1 - \xi_i^{\nu_i}}{1 - \xi_i} \right|^2$ in $\mathcal{N}(\Sigma(k_0, k_1, \dots, k_n))$. Note that by Lemma 2.7 for any α_i at least one of the exponents of q_i in each of the two Heegaard splittings is equal to 1. Lemma 1.9 shows furthermore that this is possible if and only if $\nu_i = \nu'_i$ or $\nu_i = \alpha_i - \nu'_i$. The last case is possible only for $\gamma_i = \beta_i = \pm 1 \pmod{\alpha_i}$. Hence the equation $\mathcal{N}(\Sigma(i_0, i_1, \dots, i_j)) = \mathcal{N}(\Sigma(k_0, k_1, \dots, k_n))$ implies that the conditions on the indices in Theorem 2.8 are satisfied, and the two Heegaard splittings are isotopic. \square

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